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Generalized Feedback Vertex Set Problems on Bounded-Treewidth Graphs: Chordality Is the Key to Single-Exponential Parameterized Algorithms^{*†}

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Abstract

It has long been known that FEEDBACK VERTEX SET can be solved in time $2^{O(w \log w)} n^{O(1)}$ on graphs of treewidth w , but it was only recently that this running time was improved to $2^{O(w)} n^{O(1)}$, that is, to single-exponential parameterized by treewidth. We investigate which generalizations of FEEDBACK VERTEX SET can be solved in a similar running time. Formally, for a class of graphs \mathcal{P} , BOUNDED \mathcal{P} -BLOCK VERTEX DELETION asks, given a graph G on n vertices and positive integers k and d , whether G contains a set S of at most k vertices such that each block of $G - S$ has at most d vertices and is in \mathcal{P} . Assuming that \mathcal{P} is recognizable in polynomial time and satisfies a certain natural hereditary condition, we give a sharp characterization of when single-exponential parameterized algorithms are possible for fixed values of d :

- if \mathcal{P} consists only of chordal graphs, then the problem can be solved in time $2^{O(wd^2)} n^{O(1)}$,
- if \mathcal{P} contains a graph with an induced cycle of length $\ell \geq 4$, then the problem is not solvable in time $2^{o(w \log w)} n^{O(1)}$ even for fixed $d = \ell$, unless the ETH fails.

We also study a similar problem, called BOUNDED \mathcal{P} -COMPONENT VERTEX DELETION, where the target graphs have connected components of small size instead of having blocks of small size, and present analogous results.

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1 Introduction

Treewidth is a measure of how well a graph accommodates a decomposition into a tree-like structure. In the field of parameterized complexity, many NP-hard problems have been shown to have FPT algorithms when parameterized by treewidth; for example, COLORING, VERTEX COVER, FEEDBACK VERTEX SET, and STEINER TREE. In fact, Courcelle [6] established a

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[†] The full version can be found in [5], <https://arxiv.org/abs/1704.06757>.



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meta-theorem that every problem definable in MSO_2 logic can be solved in linear time on graphs of bounded treewidth. While Courcelle's Theorem is a very general tool for obtaining algorithmic results, for specific problems dynamic programming techniques usually give algorithms where the running time $f(w)n^{\mathcal{O}(1)}$ has better dependence on treewidth w . There is some evidence that careful implementation of dynamic programming (plus maybe some additional ideas) gives optimal dependence for some problems (see, e.g., [12]).

For FEEDBACK VERTEX SET, standard dynamic programming techniques give $2^{\mathcal{O}(w \log w)} n^{\mathcal{O}(1)}$ -time algorithms and it was considered plausible that this could be the best possible running time. Hence it was a remarkable surprise when it turned out that $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$ algorithms are also possible for this problem by various techniques: Cygan et al. [7] obtained a $3^w n^{\mathcal{O}(1)}$ -time randomized algorithm by using the so-called Cut & Count technique, and Bodlaender et al. [2] showed there is a deterministic $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$ -time algorithm by using a rank-based approach and the concept of representative sets. This was also later shown in the more general setting of representative sets in matroids by Fomin et al. [11].

Generalized feedback vertex set problems. We explore the extent to which these results apply for generalizations of FEEDBACK VERTEX SET. The FEEDBACK VERTEX SET problem asks for a set S of at most k vertices such that $G - S$ is acyclic, or in other words, every block of $G - S$ is a single edge or vertex. We consider generalizations where we allow the blocks to be some other type of small graph, such as triangles, small cycles, or small cliques; these generalizations were first studied in [4]. The main result of this paper is that the existence of single-exponential algorithms is closely linked to whether the small graphs we are allowing are all chordal or not. Formally, we consider the following problem:

BOUNDED \mathcal{P} -BLOCK VERTEX DELETION

Parameter: d, w

Input: A graph G of treewidth at most w , and positive integers d and k .

Question: Is there a set S of at most k vertices in G such that each block of $G - S$ has at most d vertices and is in \mathcal{P} ?

The result of Bodlaender et al. [2] implies that when $d = 2$, BOUNDED \mathcal{P} -BLOCK VERTEX DELETION can be solved in time $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$. Our main question is for which graph classes \mathcal{P} can this problem be solved in time $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$, when we regard d as a fixed constant. A graph is *chordal* if it has no induced cycles of length at least 4. We show that if \mathcal{P} consists of only chordal graphs, then we can solve this problem in single-exponential time for fixed d .

► **Theorem 1.** *Let \mathcal{P} be a class of graphs that is block-hereditary, recognizable in polynomial time, and consists of only chordal graphs. Then BOUNDED \mathcal{P} -BLOCK VERTEX DELETION can be solved in time $2^{\mathcal{O}(wd^2)} k^2 n$ on graphs with n vertices and treewidth w .*

The condition that \mathcal{P} is block-hereditary ensures that the class of graphs with blocks in \mathcal{P} is hereditary; a formal definition is given in Section 2. We complement this result by showing that if \mathcal{P} contains a graph that is not chordal, then single-exponential algorithms are not possible (assuming ETH), even for fixed d . Note that if \mathcal{P} is block-hereditary and contains a graph that is not chordal, then this graph contains a chordless cycle on $\ell \geq 4$ vertices and consequently the cycle graph on ℓ vertices is also in \mathcal{P} .

► **Theorem 2.** *If \mathcal{P} contains the cycle graph on $\ell \geq 4$ vertices, then BOUNDED \mathcal{P} -BLOCK VERTEX DELETION is not solvable in time $2^{o(w \log w)} n^{\mathcal{O}(1)}$ on graphs of treewidth at most w even for fixed $d = \ell$, unless the ETH fails.*

Baste et al. [1] recently studied the complexity of a similar problem, where the task is to find a set of vertices whose deletion results in a graph with no minor in a given collection

of graphs \mathcal{F} , parameterized by treewidth. When $\mathcal{F} = \{C_4\}$, this is equivalent to BOUNDED \mathcal{P} -BLOCK VERTEX DELETION where $\mathcal{P} = \{K_2, K_3\}$, and the complexity they obtain in this case is consistent with our result.

Whether this lower bound of Theorem 2 is best possible when \mathcal{P} contains a cycle on $\ell \geq 4$ vertices remains open. However, as partial evidence towards this, we note that when \mathcal{P} contains all graphs, the result by Baste et al. [1] implies that that BOUNDED \mathcal{P} -BLOCK VERTEX DELETION can be solved in time $2^{\mathcal{O}(w \log w)} n^{\mathcal{O}(1)}$ when d is fixed, as the minor obstruction set \mathcal{F} consists of all of 2-connected graphs with $d + 1$ vertices.

Bounded-size components. Using a similar technique, we can obtain analogous results for a slightly simpler problem, that we call BOUNDED \mathcal{P} -COMPONENT VERTEX DELETION, where we want to remove at most k vertices such that each connected component of the resulting graph has at most d vertices and belongs to \mathcal{P} . If we have only the size constraint (i.e., \mathcal{P} contains every graph), then this problem is known as COMPONENT ORDER CONNECTIVITY [9]. Drange et al. [9] studied the parameterized complexity of a weighted variant of the COMPONENT ORDER CONNECTIVITY problem; their results imply, in particular, that COMPONENT ORDER CONNECTIVITY can be solved in time $2^{\mathcal{O}(k \log d)} n$, but is $W[1]$ -hard parameterized by only k or d . The corresponding edge-deletion problem, parameterized by treewidth, was studied by Enright and Meeks [10].

► **Theorem 3.** *Let \mathcal{P} be a class of graphs that is hereditary, recognizable in polynomial time, and consists of only chordal graphs. Then BOUNDED \mathcal{P} -COMPONENT VERTEX DELETION can be solved in time $2^{\mathcal{O}(wd^2)} k^2 n$ on graphs with n vertices and treewidth w .*

► **Theorem 4.** *If \mathcal{P} contains the cycle graph on $\ell \geq 4$ vertices, then BOUNDED \mathcal{P} -COMPONENT VERTEX DELETION is not solvable in time $2^{\mathcal{O}(w \log w)} n^{\mathcal{O}(1)}$ on graphs of treewidth at most w even for fixed $d = \ell$, unless the ETH fails.*

The result of Baste et al. [1] implies that when \mathcal{P} contains all graphs, BOUNDED \mathcal{P} -COMPONENT VERTEX DELETION can be solved in time $2^{\mathcal{O}(w \log w)} n^{\mathcal{O}(1)}$. When d is not fixed, one might ask whether BOUNDED \mathcal{P} -COMPONENT VERTEX DELETION admits an $f(w)n^{\mathcal{O}(1)}$ -time algorithm; that is, an FPT algorithm parameterized only by treewidth. We provide a negative answer: the problem is $W[1]$ -hard when \mathcal{P} contains all chordal graphs, even parameterized by both treewidth and k . Furthermore, two stronger lower bound results hold, under the assumption of the ETH.

► **Theorem 5.** *Let \mathcal{P} be a hereditary class containing all chordal graphs. Then BOUNDED \mathcal{P} -COMPONENT VERTEX DELETION is $W[1]$ -hard parameterized by the combined parameter (w, k) . Moreover, unless the ETH fails, (1) this problem has no $f(w)n^{o(w)}$ -time algorithm; and (2) it has no $f(k')n^{o(k'/\log k')}$ -time algorithm, where $k' = w + k$.*

Techniques. A pair (G, S) consisting of a graph G and a vertex subset S of G will be called a boundaried graph, and an S -block of G is a block of G containing an edge with both endpoints in S . The algorithm for BOUNDED \mathcal{P} -BLOCK VERTEX DELETION uses several lemmas on S -blocks of boundaried graphs (G, S) , which appear in Section 3. The key property is the following: (*) when we merge two boundaried graphs (G, S) and (H, S) into a graph G' , to decide whether each S -block of G' is some fixed target graph that is chordal, it is sufficient to know, for each non-trivial block B of $G[S]$ or $H[S]$, some local information about B in the S -block containing B in G or H , respectively. We think of target graphs as labeled graphs where any two vertices in the same block have distinct labels in

$\{1, \dots, d\}$, and the local information referred to in (*) is the set of labels of neighbors of B in the S -block containing B . The related result is stated as Proposition 6. This will be used to determine whether each of the S -blocks of G' is one of the target graphs in \mathcal{P} . After then, to decide whether G' is a required graph, it remains to check that the whole graph has no chordless cycle, since there is a possibility of linking two controlled blocks by a sequence of uncontrolled blocks in both sides G and H , and thus creating a chordless cycle in G' . This second part can be dealt with in a similar manner to the single-exponential time algorithm for FEEDBACK VERTEX SET, using representative-set techniques.

2 Preliminaries

We follow the terminology of Diestel [8], unless otherwise specified. A vertex v of G is a *cut vertex* if the deletion of v from G increases the number of connected components. We say G is *biconnected* if it is connected and has no cut vertices. Note that every connected graph on at most two vertices is biconnected. A *block* of G is a maximal biconnected subgraph of G . We say G is *2-connected* if it is biconnected and $|V(G)| \geq 3$. An induced cycle of length at least four is called a *chordless cycle*. A graph is *chordal* if it has no chordless cycles. For a class of graphs \mathcal{P} , a graph is called a *\mathcal{P} -block graph* if each of its blocks is in \mathcal{P} . A class \mathcal{C} of graphs is *block-hereditary* if for every $G \in \mathcal{C}$ and every biconnected induced subgraph H of G , $H \in \mathcal{C}$. For two integers d_1, d_2 with $d_1 \leq d_2$, let $[d_1, d_2]$ be the set of all integers i with $d_1 \leq i \leq d_2$, and for a positive integer, let $[d] := [1, d]$. For a function $f : X \rightarrow Y$ and $X' \subseteq X$, the function $f' : X' \rightarrow Y$ where $f'(x) = f(x)$ for all $x \in X'$ is called the *restriction* of f on X' , and is denoted $f|_{X'}$. We also say that f *extends* f' to the set X .

Block d -labeling. A *block d -labeling* of a graph G is a function $L : V(G) \rightarrow [d]$ such that for each block B of G , $L|_{V(B)}$ is an injection. If G is equipped with a block d -labeling L , then it is called a (block) *d -labeled graph*, and we call $L(v)$ the *label* of v . Two d -labeled graphs G and H are *label-isomorphic* if there is a graph isomorphism from G to H that is label preserving. For biconnected block d -labeled graphs G and H , H is *partially label-isomorphic* to G if H is label-isomorphic to the subgraph of G induced by the vertices with labels in H .

Treewidth. A *tree decomposition* of a graph G is a pair (T, \mathcal{B}) consisting of a tree T and a family $\mathcal{B} = \{B_t\}_{t \in V(T)}$ of sets $B_t \subseteq V(G)$, called *bags*, satisfying the following three conditions: (1) $V(G) = \bigcup_{t \in V(T)} B_t$, (2) for every edge uv of G , there exists a node t of T such that $u, v \in B_t$, (3) for $t_1, t_2, t_3 \in V(T)$, $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$ whenever t_2 is on the path from t_1 to t_3 in T . The *width* of a tree decomposition (T, \mathcal{B}) is $\max\{|B_t| - 1 : t \in V(T)\}$. The *treewidth* of G is the minimum width over all tree decompositions of G . A tree decomposition $(T, \mathcal{B} = \{B_t\}_{t \in V(T)})$ is *nice* if T is a rooted tree with root node r , and every node t of T is one of the following: (1) a *leaf node*: t is a leaf of T and $B_t = \emptyset$; (2) an *introduce node*: t has exactly one child t' and $B_t = B_{t'} \cup \{v\}$ for some $v \in V(G) \setminus B_{t'}$; (3) a *forget node*: t has exactly one child t' and $B_t = B_{t'} \setminus \{v\}$ for some $v \in B_{t'}$; or (4) a *join node*: t has exactly two children t_1 and t_2 , and $B_t = B_{t_1} = B_{t_2}$.

Boundaried graphs. For a graph G and $S \subseteq V(G)$, the pair (G, S) is a *boundaried graph*. When G is a d -labeled graph, we simply say that (G, S) is a *d -labeled graph*. Two d -labeled graphs (G, S) and (H, S) are said to be *compatible* if $V(G - S) \cap V(H - S) = \emptyset$, $G[S] = H[S]$, and G and H have the same labels on S . For two compatible d -labeled graphs (G, S) and (H, S) , the *sum* of two graphs $(G, S) \oplus (H, S)$ is the graph obtained from the disjoint union of

G and H by identifying each vertex in S and removing an edge if multiple edges appear. We denote by $L_G \oplus L_H$ the function from $V((G, S) \oplus (H, S))$ to $[d]$ where for $v \in V(G) \cup V(H)$, $(L_G \oplus L_H)(v) = L_G(v)$ if $v \in V(G)$ and $(L_G \oplus L_H)(v) = L_H(v)$ otherwise. For two unlabeled boundaried graphs, we define the sum in the same way, but ignoring the label condition.

A block of a graph is *non-trivial* if it has at least two vertices. For a boundaried graph (G, S) , a block B of G is called an S -block if it contains an edge of $G[S]$. Note that every non-trivial block of $G[S]$ is contained in a unique S -block of G because two distinct blocks share at most one vertex. Let (G, S) be a boundaried graph. We define $\mathbf{Aux}(G, S)$ as the bipartite boundaried graph with bipartition (C_1, C_2) and boundary C_2 such that (1) C_1 is the set of components of G , and C_2 is the set of components of $G[S]$, (2) for $C_1 \in C_1$ and $C_2 \in C_2$, $C_1 C_2 \in E(\mathbf{Aux}(G, S))$ if and only if C_2 is contained in C_1 . When (G, S) and (H, S) are two compatible d -labeled graphs, $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$ is well-defined, as G and H have the same set of components on S . For a set S and a set \mathcal{X} of subsets of S , let $\mathbf{Inc}(S, \mathcal{X})$ be the bipartite graph on the bipartition (S, \mathcal{X}) where for $v \in S$ and $X \in \mathcal{X}$, v and X are adjacent in $\mathbf{Inc}(S, \mathcal{X})$ if and only if $v \in X$. For a boundaried graph (G, S) , when \mathcal{P} is the partition of the set \mathcal{C} of components of $G[S]$ such that two components of $G[S]$ are in the same part if and only if they are in the same component of G , we denote by $\mathbf{Inc}(\mathcal{C}, \mathcal{P}) \sim \mathbf{Aux}(G, S)$.

3 Lemmas about S -blocks

We present several lemmas regarding S -blocks. For a biconnected d -labeled graph Q , a d -labeled graph (G, S) is *block-wise partially label-isomorphic to Q* if every S -block B of G is partially label-isomorphic to Q . For two compatible d -labeled graphs (G, S) and (H, S) with labelings L_G and L_H respectively, we say (G, S) and (H, S) are *block-wise Q -compatible* if

1. (G, S) and (H, S) are block-wise partially label-isomorphic to Q ; and
2. for every non-trivial block B of $G[S]$, letting B_1 and B_2 be the S -blocks of G and H that contain B , respectively, $L_G(N_{B_1}(V(B)) \setminus S) \cap L_H(N_{B_2}(V(B)) \setminus S) = \emptyset$, and, for $\ell_1 \in L_G(N_{B_1}(V(B)) \setminus S)$ and $\ell_2 \in L_H(N_{B_2}(V(B)) \setminus S)$, the vertices in Q with labels ℓ_1 and ℓ_2 are not adjacent.

We describe sufficient conditions for when, given a chordal labeled graph Q , the sum of two given labeled graphs (G, S) and (H, S) , each partially label-isomorphic to Q , is also partially label-isomorphic to Q .

► **Proposition 6.** *Let Q be a biconnected d -labeled chordal graph. Let (G, S) and (H, S) be two block-wise Q -compatible d -labeled graphs such that $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$ has no cycles. Then $(G, S) \oplus (H, S)$ is block-wise partially label-isomorphic to Q .*

We use the following essential property of chordal graphs.

► **Lemma 7.** *Let F be a connected graph and let Q be a connected chordal graph. Let $\mu : V(F) \rightarrow V(Q)$ be a function such that for every induced path $p_1 \cdots p_m$ in F of length at most two, $\mu(p_1), \dots, \mu(p_m)$ are pairwise distinct and $\mu(p_1) \cdots \mu(p_m)$ is an induced path of Q . Then μ is an injection and preserves the adjacency relation.*

► **Lemma 8.** *Let (G, S) and (H, S) be two compatible d -labeled graphs such that $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$ has no cycles. (1) If F is an S -block of $(G, S) \oplus (H, S)$ and uv is an edge in F , then uv is contained in some S -block of G or H . (2) Suppose each S -block of G or H is chordal. If F is an S -block of $(G, S) \oplus (H, S)$ and uvw is an induced path in F such that u and w are not contained in the same S -block of G or H , then $v \in S$, and there is an induced path $q_1 q_2 \cdots q_\ell$ from $u = q_1$ to $w = q_\ell$ in $F - v$ such that each q_i is a neighbor of v .*

Proof of Proposition 6. Let F be an S -block of $(G, S) \oplus (H, S)$. Let L_G and L_H be labelings of G and H , respectively, and let $L := L_G \oplus L_H$. We may assume $|V(F)| \geq 3$. By Lemma 8, every edge of F is contained in some S -block of G or H . Thus, for $uv \in E(F)$, we have $L(u) \neq L(v)$ and the vertices with labels $L(u)$ and $L(v)$ are adjacent in Q . Moreover, since (G, S) and (H, S) are block-wise partially label-isomorphic to Q , we have $L(V(F)) \subseteq L_Q(V(Q))$. Let $\mu : V(F) \rightarrow V(Q)$ such that for each $v \in V(F)$, $L(v) = L_Q(\mu(v))$.

To apply Lemma 7, it is sufficient to prove that if uvw is an induced path in F , then $L(u) \neq L(w)$ and $\mu(u)\mu(v)\mu(w)$ is an induced path in Q . Since (G, S) and (H, S) are block-wise partially label-isomorphic to Q , if all of u, v, w are contained in an S -block of G or H , then it follows from the given condition. We may assume u and w are not contained in the same S -block of G or H . Then by (2) of Lemma 8, $v \in S$, and there is an induced path $q_1 q_2 \cdots q_\ell$ from $u = q_1$ to $w = q_\ell$ in $F - v$ such that each q_i is a neighbor of v .

We show that for $i \in \{1, \dots, \ell - 2\}$, $L(q_i), L(q_{i+1}), L(q_{i+2})$ are pairwise distinct, and $\mu(q_i)\mu(q_{i+1})\mu(q_{i+2})$ is an induced path of Q . If all of q_i, q_{i+1}, q_{i+2} are contained in G or H , then they are contained in the same S -block as v , and the claim follows. We may assume q_i and q_{i+2} are in distinct graphs of $G - S$ and $H - S$. Then the S -block containing q_i, q_{i+1}, v and the S -block containing q_{i+1}, q_{i+2}, v share the edge $q_{i+1}v$. Since (G, S) and (H, S) are block-wise Q -compatible, $L(q_i) \neq L(q_{i+2})$ and $\mu(q_i)$ is not adjacent to $\mu(q_{i+2})$ in Q .

We verify that $\mu(q_1)\mu(q_2) \cdots \mu(q_\ell)$ is an induced path of Q . Suppose this is false, and choose $i_1, i_2 \in \{1, 2, \dots, \ell\}$ with $i_2 - i_1 > 1$ and minimum $i_2 - i_1$ such that $\mu(q_{i_1})$ is adjacent to $\mu(q_{i_2})$ in Q . By minimality, $\mu(q_{i_1}) \cdots \mu(q_{i_2-1})$ and $\mu(q_{i_1+1}) \cdots \mu(q_{i_2})$ are induced paths and have length at least 2. Thus $\mu(q_{i_1}) \cdots \mu(q_{i_2})$ is an induced cycle of length at least 4, contradicting the assumption that Q is chordal. Therefore, $\mu(q_1)\mu(q_2) \cdots \mu(q_\ell)$ is an induced path of Q , and, in particular, $L(u) \neq L(w)$ and $\mu(u)$ and $\mu(w)$ are not adjacent in Q , as required. By Lemma 7, we conclude that F is partially label-isomorphic to Q . ◀

Using Lemma 8, we can also prove the following.

► **Lemma 9.** Let A be a set, let (G, S) and (H, S) be two compatible d -labeled graphs, and let \mathcal{B} be the set of non-trivial blocks in $G[S]$. Suppose $g : \mathcal{B} \rightarrow A$ is a function where each S -block of G or H is chordal, $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$ has no cycles, and for every $B_1, B_2 \in \mathcal{B}$ where B_1 and B_2 are contained in an S -block of G or H , $g(B_1) = g(B_2)$. If F is an S -block of $(G, S) \oplus (H, S)$ and $B_1, B_2 \in \mathcal{B}$ where $V(B_1), V(B_2) \subseteq V(F)$, then $g(B_1) = g(B_2)$.

► **Proposition 10.** Let (G, S) and (H, S) be two compatible d -labeled graphs such that every S -block of $(G, S) \oplus (H, S)$ is chordal. Then $(G, S) \oplus (H, S)$ is chordal if and only if $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$ has no cycles.

Proof. We briefly sketch the proof of one direction. Suppose that $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$ has a cycle $C_1 - A_1 - C_2 - A_2 - \cdots - C_n - A_n - C_1$ where C_1, \dots, C_n are components of $G[S]$. For each $i \in \{1, \dots, n\}$, let P_i be the shortest path from C_i to C_{i+1} in A_i , and let v_i, w_i be the end vertices of P_i where $v_i \in V(C_i)$ and $w_i \in V(C_{i+1})$. Let Q_i be the shortest path from w_i to v_{i+1} in C_{i+1} . We may assume $n \geq 3$; it is easy when $n = 2$. Then $v_1 P_1 - Q_1 - P_2 - Q_2 - \cdots - P_n - Q_n v_1$ is a cycle in $(G, S) \oplus (H, S)$, but is not necessarily a chordless cycle. We claim that it contains a chordless cycle. Let x be the vertex following v_2 in P_2 , and let y be the vertex preceding w_n in P_n . Take a shortest path P from x to y in the path $y - Q_n - P_1 - Q_1 - x$. Clearly P has length at least 2, as x and y are contained in distinct connected components of G or H . Also, every internal vertex of P has no neighbors in the other path of the cycle $v_1 P_1 - Q_1 - P_2 - Q_2 - \cdots - P_n - Q_n v_1$ between x and y . So, if we take a shortest path P' from x to y along the other part of the cycle $v_1 P_1 - Q_1 - P_2 - Q_2 - \cdots - P_n - Q_n v_1$, then $P \cup P'$ is a chordless cycle. ◀

4 Bounded \mathcal{P} -Block Vertex Deletion

We prove Theorem 1. We first focus on S -blocks of boundaried graphs (G, S) . For each non-trivial block of $G[S]$, we guess its final shape, as a d -labeled biconnected graph, and store the labelings of the vertices and their neighbors in the S -block of G containing it. Collectively, we call this information a *characteristic* of (G, S) . Using characteristics, we control S -blocks in $(G, S) \oplus (H, S)$, where (H, S) is a compatible d -labeled graph. By the previous step, we may assume that every S -block of $(G, S) \oplus (H, S)$ is in \mathcal{P} and has at most d vertices. Note that $(G, S) \oplus (H, S)$ still may have a chordless cycle. By Proposition 10, if we assume that every S -block of $(G, S) \oplus (H, S)$ is in \mathcal{P} , then $(G, S) \oplus (H, S)$ is chordal if and only if $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$ has no cycles. So, instead of keeping $\mathbf{Aux}(G, S)$, we store the corresponding partition of the set of components of $G[S]$.

For convenience, we fix an integer $d \geq 2$ and a class \mathcal{P} of graphs that is block-hereditary, recognizable in polynomial time, and consists of only chordal graphs. Let \mathcal{U}_d be the set of all d -labeled biconnected \mathcal{P} -block graphs, where each H in \mathcal{U}_d has labeling L_H . For a boundaried graph (G, S) , we denote by $\text{Block}(G, S)$ the set of all non-trivial blocks in $G[S]$.

For a d -labeled graph (G, S) with a labeling L , a *characteristic* of (G, S) is a pair (g, h) of functions $g : \text{Block}(G, S) \rightarrow \mathcal{U}_d$ and $h : \text{Block}(G, S) \rightarrow 2^{[d]}$ satisfying the following, for each $B \in \text{Block}(G, S)$ and the unique S -block H of G containing B ,

1. (label-isomorphic condition) H is partially label-isomorphic to $g(B)$;
2. (coincidence condition) for every $B' \in \text{Block}(G, S)$ with $V(B') \subseteq V(H)$, $g(B') = g(B)$;
3. (neighborhood condition) $h(B) = L(N_H(V(B)) \setminus S)$; and
4. (complete condition) for every w where $w \in V(H) \setminus S$ or $\{w\} = V(H) \cap V(C)$ for some component C of $G[S]$, $H[N_H[w]]$ is label-isomorphic to $g(B)[N_{g(B)}[z]]$ where z is the vertex in $g(B)$ with label $L(w)$.

We say that the sum $(G, S) \oplus (H, S)$ *respects* (g, h) if for each $B \in \text{Block}(G, S)$, the S -block of $(G, S) \oplus (H, S)$ containing B is label-isomorphic to $g(B)$. The following is the main combinatorial result regarding characteristics.

► **Theorem 11.** *Let (G_1, S) , (G_2, S) , (H, S) be d -labeled \mathcal{P} -block graphs such that each (G_i, S) is compatible with (H, S) , (G_1, S) and (G_2, S) have the same characteristic (g, h) , and $\mathbf{Aux}(G_2, S) \oplus \mathbf{Aux}(H, S)$ has no cycles. If $(G_1, S) \oplus (H, S)$ is a d -labeled \mathcal{P} -block graph that respects (g, h) , then $(G_2, S) \oplus (H, S)$ is a d -labeled \mathcal{P} -block graph that respects (g, h) .*

Proof. We show $(G_2, S) \oplus (H, S)$ respects (g, h) . Choose a non-trivial block B of $G_2[S]$, let $Q := g(B)$, let F be the S -block of $(G_2, S) \oplus (H, S)$ containing B , L_F be the function from $V(F)$ to $[d]$ that sends each vertex to its label from G_2 or H , and L_Q be the labeling of Q .

We claim that F is label-isomorphic to Q . We regard F as the sum of $(F \cap G_2, V(F) \cap S)$ and $(F \cap H, V(F) \cap S)$ and verify the conditions of Proposition 6. Using Lemma 9, for every $B' \in \text{Block}(G_2, S)$ with $V(B') \subseteq V(F)$, $g(B') = Q$. We also observe that $\mathbf{Aux}(F \cap G_2, S_F) \oplus \mathbf{Aux}(F \cap H, S_F)$ has no cycles as $\mathbf{Aux}(G_2, S) \oplus \mathbf{Aux}(H, S)$ has no cycles. Since (g, h) is a characteristic of (G_2, S) and $(G_1, S) \oplus (H, S)$ respects (g, h) , we can confirm that both $F \cap G$ and $F \cap H$ are block-wise partially label-isomorphic to Q . The second condition of being block-wise Q -compatible follows from the fact that (G_1, S) and (G_2, S) have the same characteristic (g, h) . Thus, $F \cap G_2$ and $F \cap H$ are block-wise Q -compatible, and this implies that F is partially label-isomorphic to Q by Proposition 6. By the ‘complete condition’ of a characteristic, we can show that $L_Q(V(Q)) \subseteq L_F(V(F))$, so F is label-isomorphic to Q .

Lastly, we can confirm that $(G_2, S) \oplus (H, S)$ is a d -labeled \mathcal{P} -block graph by showing that every non S -block of $(G_2, S) \oplus (H, S)$ is fully contained in G_2 or H . We can argue this using the fact that $(G_2, S) \oplus (H, S)$ is chordal, which is implied by Proposition 10. ◀

Proof of Theorem 1. We obtain a nice tree decomposition $(T, \mathcal{B} = \{B_t\}_{t \in V(T)})$ of G with root node r and width at most $5w + 4$ in time $\mathcal{O}(c^w \cdot n)$ for some constant c using the approximation algorithm by Bodlaender et al. [3]. For $t \in V(T)$, let G_t be the subgraph of G induced by the union of all bags $B_{t'}$ where t' is a descendant of t . Let $\text{Comp}(t, X)$ be the set of all components of $G[B_t \setminus X]$, and $\text{Part}(t, X)$ be the set of all partitions of $\text{Comp}(t, X)$.

For each node t of T , $X \subseteq B_t$, and a function $L : B_t \setminus X \rightarrow [d]$, we define $\mathcal{F}(t, X, L)$ as the set of all pairs (g, h) consisting of functions $g : \text{Block}(t, X) \rightarrow \mathcal{U}_d$ and $h : \text{Block}(t, X) \rightarrow 2^{[d]}$. We say that (g, h) is *valid*, if (1) L is a d -labeling of $G[B_t \setminus X]$, (2) for each $B \in \text{Block}(t, X)$, B is partially label-isomorphic to $g(B)$, and (3) for each $B \in \text{Block}(t, X)$, $L(V(B)) \cap h(B) = \emptyset$. For $i \in \{0, 1, \dots, k\}$ and $(g, h) \in \mathcal{F}(t, X, L)$, let $c[t, (X, L, i, (g, h))]$ be the family of all partitions $\mathcal{X} \in \text{Part}(t, X)$ satisfying the following property: there exist $S \subseteq V(G_t) \setminus B_t$ with $|S| = i$ and a d -labeling L' of $G_t - (X \cup S)$ where (1) $L = L'|_{B_t \setminus X}$, (2) $G_t - (X \cup S)$ is a \mathcal{P} -block graph, (3) (g, h) is a characteristic of $(G_t - (X \cup S), B_t \setminus X)$, and (4) $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{X}) \sim \mathbf{Aux}(G_t - (X \cup S), B_t \setminus X)$. Such a pair (S, L') is a *partial solution* with respect to \mathcal{X} .

The main idea is that instead of fully computing $c[t, M]$ for $M = (X, L, i, (g, h))$, we recursively enumerate a set $r[t, M]$ that may represent partial solutions for $c[t, M]$. Formally, for a subset $r[t, M] \subseteq c[t, M]$, we denote $r[t, M] \equiv c[t, M]$ if for every $\mathcal{X} \in c[t, M]$ and a partial solution (S, L') with respect to \mathcal{X} and $S_{\text{out}} \subseteq V(G) \setminus V(G_t)$ where $G - (S \cup X \cup S_{\text{out}})$ is a d -labeled \mathcal{P} -block graph respecting (g, h) , there exists $\mathcal{X}_1 \in r[t, M]$ and a partial solution (S', L'') with respect to \mathcal{X}_1 such that $G - (S' \cup X \cup S_{\text{out}})$ is a d -labeled \mathcal{P} -block graph respecting (g, h) . By the definition of $r[t, M]$, the problem is a YES-instance if and only if there exists $(X, L, i, (g, h))$ for the root node r with $|X| + i \leq k$ such that $r[r, (X, L, i, (g, h))] \neq \emptyset$.

Whenever we update $r[t, M]$, we confirm that $|r[t, M]| \leq w \cdot 2^{w-1}$. This will be the application of the representative set technique developed by Bodlaender et al. [2]. For a set S and a set \mathcal{A} of partitions of S , a subset \mathcal{A}' of \mathcal{A} is called a *representative set* if for every $\mathcal{X}_1 \in \mathcal{A}$ and every partition \mathcal{Y} of S where $\mathbf{Inc}(S, \mathcal{X}_1 \cup \mathcal{Y})$ has no cycles, there exists a partition $\mathcal{X}_2 \in \mathcal{A}'$ such that $\mathbf{Inc}(S, \mathcal{X}_2 \cup \mathcal{Y})$ has no cycles.

► **Proposition 12.** *Given a family \mathcal{A} of partitions of a set S , one can output a representative set of \mathcal{A} of size at most $|S| \cdot 2^{|S|-1}$ in time $\mathcal{A}^{\mathcal{O}(1)} 2^{\mathcal{O}(|S|)}$.*

We sketch how to update families $r[t, M]$ when t is an introduce node with child node t' . We may assume (g, h) is valid, otherwise $c[t, M] = \emptyset$.

Let v be the vertex in $B_t \setminus B_{t'}$. If $v \in X$, then $G_t - X = G_{t'} - (X \setminus \{v\})$ and $B_t \setminus X = B_{t'} \setminus (X \setminus \{v\})$. Thus, we can set $r[t, M] := r[t', (X \setminus \{v\}, L, i, (g, h))]$. We assume $v \notin X$, and let $L_{\text{res}} := L|_{B_{t'} \setminus X}$. For $(g, h) \in \mathcal{F}(t, X, L)$, a pair $(g', h') \in \mathcal{F}(t', X, L_{\text{res}})$ is called the *restriction* of (g, h) if (1) for $B_1 \in \text{Block}(t', X)$ and $B_2 \in \text{Block}(t, X)$ with $V(B_1) \subseteq V(B_2)$, $g'(B_1) = g(B_2)$, and if $v \in V(B_2)$, then every vertex in $g'(B_1)$ with label in $h'(B_1)$ is not adjacent to the vertex in $g'(B_1)$ with label $L(v)$, (2) for $B_1 \in \text{Block}(t', X)$ and $B_2 \in \text{Block}(t, X)$ with $V(B_1) \subseteq V(B_2)$ and $v \notin V(B_2)$, $h'(B_1) = h(B_2)$, and (3) for $B_2 \in \text{Block}(t, X)$ containing v , $h(B_2) = \bigcup_{B_1 \in \text{Block}(t', X), V(B_1) \subseteq V(B_2)} h(B_1)$.

► **Claim 13.** *For $\mathcal{X} \in \text{Part}(t, X)$, $\mathcal{X} \in c[t, M]$ if and only if there exist a restriction (g', h') of (g, h) and $\mathcal{Y} \in c[t', (X, L_{\text{res}}, i, (g', h'))]$ such that (1) v has neighbors on at most one component in each part of \mathcal{Y} , and (2) if v has at least one neighbor in $G[B_t \setminus X]$, then \mathcal{X} is the partition obtained from \mathcal{Y} by, for parts Y_1, \dots, Y_m of \mathcal{Y} containing components having a neighbor of v , removing all of Y_1, \dots, Y_m and adding a part that consists of all components of $G[B_t \setminus X]$ not contained in parts of $\mathcal{Y} \setminus \{Y_1, \dots, Y_m\}$; and otherwise, $\mathcal{X} = \mathcal{Y} \cup \{\{v\}\}$.*

We update $r[t, M]$ as follows. Set $\mathcal{K} := \emptyset$. For a pair of functions (g', h') , we test whether (g', h') is a restriction of (g, h) . Assume (g', h') is a restriction of (g, h) . For each $\mathcal{Y} \in r[t', (X, L_{res}, i, (g', h'))]$, we check the two conditions for (g', h') and \mathcal{Y} in Claim 13, and if they are satisfied, then add the set \mathcal{X} described in Claim 13 to \mathcal{K} ; otherwise, skip it. The whole procedure can be done in time $2^{\mathcal{O}(wd^2)}$. After we do this for all possible candidates, we take a representative set of \mathcal{K} using Proposition 12, and assign the resulting set to $r[t, M]$.

We claim that $r[t, M] \equiv c[t, M]$. Let $G_{out} := G - (V(G_t) \setminus B_t)$, $\mathcal{X} \in c[t, M]$, and (S, L') be a partial solution with respect to \mathcal{X} , and suppose there exists $S_{out} \subseteq V(G) \setminus V(G_t)$ where $(G_t - (X \cup S), B_t \setminus X) \oplus (G_{out} - (X \cup S_{out}), B_t \setminus X)$ is a d -labeled \mathcal{P} -block graph respecting (g, h) . Every $(B_{t'} \setminus X)$ -block of $G - (S \cup X \cup S_{out})$ is chordal as such a block is a $(B_t \setminus X)$ -block of $G - (S \cup X \cup S_{out})$. Since $G - (S \cup X \cup S_{out})$ is chordal, by Proposition 10, $\mathbf{Aux}(G_{t'} - (X \cup S), B_{t'} \setminus X) \oplus \mathbf{Aux}(G_{out} - (X \cup S_{out}), B_{t'} \setminus X)$ has no cycles. Let $M_{res} := (X, L_{res}, i, (g', h'))$. As $r[t', M_{res}] \equiv c[t', M_{res}]$, there exist $\mathcal{Y} \in r[t', M_{res}]$ and a partial solution (S', L'') with respect to \mathcal{Y} such that $\mathbf{Inc}(\text{Comp}(t', X), \mathcal{Y}) \sim \mathbf{Aux}(G_{t'} - (X \cup S'), B_{t'} \setminus X)$ has no cycles. By Theorem 11, $G - (S' \cup X \cup S_{out})$ is a d -labeled \mathcal{P} -block graph respecting (g, h) .

By the procedure, \mathcal{X}_1 where $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{X}_1) \sim \mathbf{Aux}(G_t - (X \cup S'), B_t \setminus X)$ is added to \mathcal{K} . And there exist $\mathcal{X}_2 \in r[t, M]$ and a partial solution (S'', L''') with respect to \mathcal{X}_2 such that $G - (S'' \cup X \cup S_{out})$ is a d -labeled \mathcal{P} -block graph. Thus, $r[t, M] \equiv c[t, M]$.

Total running time. We denote $|V(G)|$ by n . Note that the number of nodes in T is $\mathcal{O}(wn)$. For fixed $t \in V(T)$, there are at most 2^{w+1} possible choices for $X \subseteq B_t$, and for fixed $X \subseteq B_t$, there are at most d^{w+1} possible functions L . Furthermore, the size of $\mathcal{F}(t, X, L)$ is bounded by $2^{\mathcal{O}(wd^2)}$. Thus, there are $\mathcal{O}(n \cdot k \cdot \max(2, d)^{w+1} \cdot 2^{\mathcal{O}(wd^2)})$ tables. In summary, the algorithm runs in time $\mathcal{O}(n \cdot k \cdot \max(2, d)^{w+1} \cdot 2^{\mathcal{O}(wd^2)} \cdot k) = 2^{\mathcal{O}(wd^2)} k^2 n$. ◀

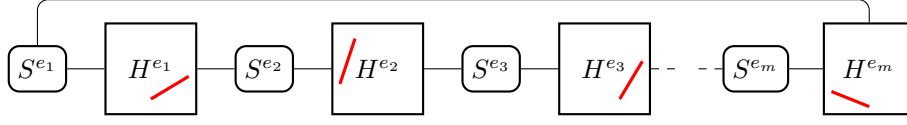
5 Lower bound for fixed d

We showed that BOUNDED \mathcal{P} -COMPONENT VERTEX DELETION and BOUNDED \mathcal{P} -BLOCK VERTEX DELETION admit single-exponential time algorithms parameterized by treewidth, whenever \mathcal{P} is a class of chordal graphs. We now establish that, assuming the ETH, this is no longer the case when \mathcal{P} contains a graph that is not chordal.

In the $k \times k$ INDEPENDENT SET problem, one is given a graph $G = ([k] \times [k], E)$ over the k^2 vertices of a k -by- k grid. We denote by $\langle i, j \rangle$ with $i, j \in [k]$ the vertex of G in the i -th row and j -th column. The goal is to find an independent set of size k in G that contains exactly one vertex in each row. The PERMUTATION $k \times k$ INDEPENDENT SET problem is similar but with the additional constraint that the independent set should also contain exactly one vertex per column.

► **Theorem 14.** *If \mathcal{P} contains the cycle graph on $\ell \geq 4$ vertices, then BOUNDED \mathcal{P} -COMPONENT VERTEX DELETION, or BOUNDED \mathcal{P} -BLOCK VERTEX DELETION, is not solvable in time $2^{o(w \log w)} n^{\mathcal{O}(1)}$ on graphs of treewidth at most w even for fixed $d = \ell$, unless the ETH fails.*

Proof. To prove this theorem, we reduce from PERMUTATION $k \times k$ INDEPENDENT SET which, like PERMUTATION $k \times k$ CLIQUE, cannot be solved in time $2^{o(k \log k)} k^{\mathcal{O}(1)}$ unless the ETH fails [13]. Let $G = ([k] \times [k], E)$ be an instance of PERMUTATION $k \times k$ INDEPENDENT SET. We assume that $\forall h, i, j \in [k]$ with $h \neq i$, $\langle i, j \rangle \langle h, j \rangle \in E$. Adding these edges does not change the YES- and NO-instances, but has the virtue of making PERMUTATION $k \times k$ INDEPENDENT SET equivalent to $k \times k$ INDEPENDENT SET. We also assume that $\forall h, i, j \in [k]$, $\langle i, j \rangle \langle i, h \rangle \notin E$,



■ **Figure 1** A high-level schematic of G' and G'' . The H^{e_i} s only differ by a constant number of edges (in red/light gray) that encode their edge e_i of G .

since at most one of $\langle i, j \rangle$ and $\langle i, h \rangle$ can be in a given solution. Let $m := |E| = \mathcal{O}(k^4)$ be the number of edges of G .

Outline. We build two graphs $G' = (V', E')$ and $G'' = (V', E'')$ with treewidth at most $(3d+4)k+6d-5 = \mathcal{O}(k)$, and $((3d-2)k^2+2k)m$ vertices, where the following are equivalent:

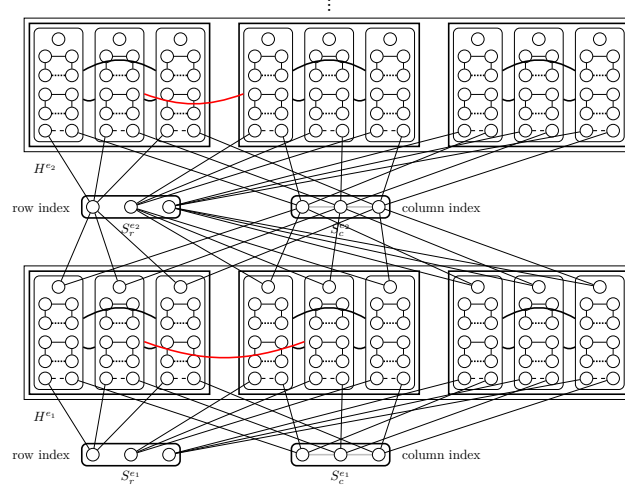
1. G has an independent set of size k with one vertex per row of G .
2. There is a set $S \subseteq V'$ of size at most $(3d-2)k(k-1)m$ such that each connected component of $G' - S$ has size at most d and belongs to \mathcal{P} .
3. There is a set $S \subseteq V'$ of size at most $(3d-2)k(k-1)m$ such that each block of $G'' - S$ has size at most d and belongs to \mathcal{P} .

The overall construction of G' and G'' will display m *almost* copies of the encoding of an *edgeless* G arranged in a cycle. Each copy embeds one distinct edge of G . The point of having the information of G distilled edge by edge in G' and G'' is to control the treewidth. This general idea originates from a paper of Lokshtanov et al. [12].

Construction. We first describe G' . As a slight abuse of notation, a gadget (and, more generally, a subpart of the construction) may refer to either a subset of vertices or to an induced subgraph. For each $e = \langle i^e, j^e \rangle \langle i'^e, j'^e \rangle \in E$, we detail the internal construction of H^e and S^e of Figure 1 and how they are linked to one another. Each vertex $v = \langle i, j \rangle$ of G is represented by a gadget $H^e(v)$ on $3d-2$ vertices in G' : a path on $d-3$ vertices whose endpoints are v_{-a}^e and v_{-b}^e , an isolated vertex v_+^e , and two disjoint cycles of length d . Observe that if $d=4$, then v_{-a}^e and v_{-b}^e is the same vertex. We add all the edges between $H^e(\langle i, j \rangle)$ and $H^e(\langle i, j' \rangle)$ for $i, j, j' \in [k]$ with $j \neq j'$. We also add all the edges between $H^e(\langle i^e, j^e \rangle)$ and $H^e(\langle i'^e, j'^e \rangle)$. We call H^e the graph induced by the union of every $H^e(v)$, for $v \in V(G)$. The *row/column selector* gadget S^e consists of a set S_r^e of k vertices with one vertex r_i^e for each row index $i \in [k]$, and a set S_c^e of k vertices with one vertex c_j^e for each column index $j \in [k]$. The gadget S^e forms an independent set of size $2k$. We arbitrarily number the edges of G : e_1, e_2, \dots, e_m . For each $h \in [m]$ and $v = \langle i, j \rangle \in V$, we link $v_{-a}^{e_h}$ to $r_i^{e_h}$ (the row index of v) and $v_{-b}^{e_h}$ to $c_j^{e_h}$ (the column index of v). We also link, for every $h \in [m-1]$, $v_+^{e_h}$ to $r_i^{e_{h+1}}$ and to $c_j^{e_{h+1}}$, and $v_+^{e_m}$ to $r_i^{e_1}$ and to $c_j^{e_1}$. That concludes the construction (see Figure 2). To obtain G'' from G' , we add the edges $c_j^{e_h} c_{j+1}^{e_h}$ for every $h \in [m]$ and $j \in [k-1]$. We ask for a deletion set S of size $s := (3d-2)k(k-1)m$.

Treewidth of G' and G'' . For any edge $e \in E$, we set $H(e) := H^e(\langle i^e, j^e \rangle) \cup H^e(\langle i'^e, j'^e \rangle)$. For any $i \in [m-1]$, we set $\tilde{S}_i := S^{e_1} \cup S^{e_i} \cup S^{e_{i+1}}$, and $\tilde{S}_m := S^{e_1} \cup S^{e_m}$. For each $e \in E$, and $i \in [k]$, $H^e(i)$ denotes the union of the $H^e(v)$ for all vertices v of the i -th row. Here is a path decomposition of G' and G'' :

$$\begin{aligned} \tilde{S}_1 \cup H(e_1) \cup H^{e_1}(1) &\rightarrow \tilde{S}_1 \cup H(e_1) \cup H^{e_1}(2) \rightarrow \dots \rightarrow \tilde{S}_1 \cup H(e_1) \cup H^{e_1}(k) \rightarrow \\ &\vdots \\ \tilde{S}_m \cup H(e_m) \cup H^{e_m}(1) &\rightarrow \tilde{S}_m \cup H(e_m) \cup H^{e_m}(2) \rightarrow \dots \rightarrow \tilde{S}_m \cup H(e_m) \cup H^{e_m}(k). \end{aligned}$$



■ **Figure 2** The overall picture of G' and G'' with $k = 3$. Dotted edges are subdivided $d - 4$ times; if $d = 4$, they are simply edges. Dashed edges are subdivided $d - 5$ times; if $d = 4$, the two endpoints are in fact a single vertex. Edges between two boxes link each vertex of one box to each vertex of the other box. The gray edges in the column selectors $S_c^{e_h}$ are only present in G'' .

As, for any $h \in [m]$, $|\tilde{S}_h| \leq 6k$, $|H(e_h)| = 2(3d - 2)$, and $|H^{e_h}(i)| \leq (3d - 2)k$ for any $i \in [k]$, the size of a bag is bounded by $\max_{h \in [m], i \in [k]} |\tilde{S}_h \cup H(e_h) \cup H^{e_h}(i)| \leq 6k + 2(3d - 2) + (3d - 2)k = (3d + 4)k + 6d - 4$.

Correctness. If there is an independent set I of size k in G , a solution to a BOUNDED \mathcal{P} -COMPONENT VERTEX DELETION or BOUNDED \mathcal{P} -BLOCK VERTEX DELETION instance can be obtained by deleting from each H^e every $H^e(v)$ such that $v \notin I$.

We show that $2 \Rightarrow 1$ and $3 \Rightarrow 1$. We assume that there is a set $S \subseteq V'$ of size at most s such that all the blocks of $G'' - S$ (resp. $G' - S$) have size at most d . We note that this corresponds to assuming condition 3 (resp. a weaker assumption than condition 2) holds. We show that there are at most $3d - 2$ vertices of $H^e(i)$ remaining in $G'' - S$ (or $G' - S$). Assume, for the sake of contradiction, that $H^e(i) - S$ contains at least $3d - 1$ vertices. Observe that $H^e(i) - S$ cannot contain at least one vertex from three distinct $H^e(u)$, $H^e(v)$, and $H^e(w)$ (with u, v and w in the i -th row of G), since then $H^e(i) - S$ would be 2-connected (and of size $> d$). For the same reason, $H^e(i) - S$ cannot contain at least two vertices in $H^e(u)$ and at least two vertices in another $H^e(v)$. Therefore, the only way of fitting $3d - 1$ vertices in $H^e(i) - S$ is the $3d - 2$ vertices of an $H^e(u)$ plus one vertex from some other $H^e(v)$. But then, this vertex of $H^e(v)$ would form, together with one C_d of $H^e(u)$, a 2-connected subgraph of $G'' - S$ (or $G' - S$) of size $d + 1$. Now, we know that $|H^e(i) \cap S| \geq (3d - 2)(k - 1)$. As there are precisely mk sets $H^e(i)$ in G' (and they are disjoint), it further holds that $|H^e(i) \cap S| = (3d - 2)(k - 1)$, since otherwise S would contain strictly more than $s = (3d - 2)k(k - 1)m$ vertices. Thus, $H^e(i) - S$ contains exactly $3d - 2$ vertices. By the previous remarks, $H^e(i) - S$ can only consist of the $3d - 2$ vertices of the same $H^e(u)$ or $3d - 3$ vertices of $H^e(u)$ plus one vertex from another $H^e(v)$. In fact, the latter case is not possible, since the vertex of $H^e(v)$ would form, with at least one remaining C_d of the $3d - 3$ vertices of $H^e(u)$, a 2-connected subgraph of $G'' - S$ (or $G' - S$) of size $d + 1$. This is why we needed two disjoint C_d s in the construction instead of just one. So far, we have proved that, assuming condition 2 or condition 3 holds, for any $e \in E$ and $i \in [k]$, $H^e(i) \cap S = H^e(v_{i,e})$ for some vertex $v_{i,e}$ of the i -th row of G , and for any $e \in E$, $S^e \cap S = \emptyset$.

In what follows, we show that $v_{i,e}$ does not depend on e . Formally, we want to show that there is a v_i such that, for any $e \in E$, $v_{i,e} = v_i$. Observe that it is enough to derive that, for any $h \in [m]$, $v_{i,e_h} = v_{i,e_{h+1}}$ (with $e_{m+1} = e_1$). Let $j \in [k]$ (resp. $j' \in [k]$) be the column of v_{i,e_h} (resp. $v_{i,e_{h+1}}$) in G . We first assume condition 2 holds. For any $h \in [m]$, $v_{i,e_h}^{e_h}$, $r_i^{e_{h+1}}$, $c_{j'}^{e_{h+1}}$, $c_j^{e_{h+1}}$ plus the path $P_{v_{i,e_{h+1}-a}^{e_{h+1}} v_{i,e_{h+1}-b}^{e_{h+1}}}$ (between $v_{i,e_{h+1}-a}^{e_{h+1}}$ and $v_{i,e_{h+1}-b}^{e_{h+1}}$) induces a path (in particular, a connected subgraph) of size $d+1$ in $G'' - S$, unless $j = j'$ (with $e_{m+1} = e_1$). Therefore, $j = j'$. As v_{i,e_h} and $v_{i,e_{h+1}}$ have the same column j and the same row i in G , $v_{i,e_h} = v_{i,e_{h+1}}$. Showing the same property under 3 is done similarly. We can now safely define $v_i := v_{i,e}$ and conclude by proving that $\{v_1, v_2, \dots, v_k\}$ is a clique. ◀

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